Combinatorica 27 (3) (2007) 383–398
DOI: 10.1007/s00493-007-2131-x

COMBINATORICA
Bolyai Society – Springer-Verlag

MATCHINGS OF CYCLES AND PATHS IN DIRECTED GRAPHS*

GYULA PAP†, LÁSZLÓ SZEGŐ‡

Received March 30, 2004

In this paper we present a Berge–Tutte-type theorem for a matching problem in directed graphs. This extends the maximum matching problem in undirected graphs, the maximum even factor problem in weakly symmetric directed graphs proposed by W. H. Cunningham and J. F. Geelen in [6], and a packing problem for cycles and edges in undirected graphs. We show an Edmonds–Gallai-type structural description of a canonical set attaining the minimum in the formula. We also give a generalization of the matching matroid to this concept.

1. Introduction

The aim of this paper is to present a common generalization of two min-max relations on packings of subgraphs, we also derive the analogue of the Edmonds–Gallai decomposition, and show a generalization of the matching matroid. The first special case is the maximum even factor problem, where we have to pack a node-disjoint union of directed paths and directed even cycles into a directed graph. The problem was proposed by W. H. Cunningham and J. F. Geelen in [6] where they pointed out that the problem is polynomial-time solvable for so-called weakly symmetric graphs. A simple

Mathematics Subject Classification (2000): 05C70

* Research supported by the Hungarian National Foundation for Scientific Research Grant, OTKA T037547 and by European MCRTN Adonet, Contract Grant No. 504438.

† The author is supported by the Egerváry Research Group of the Hungarian Academy of Sciences.

‡ The author is a member of the Egerváry Research Group (EGRES).

min-max formula was given by the authors [17]. The second special case is the following special case of the so-called hypomatching problem in an undirected graph. Given a family of "allowed odd cycles", maximize the number of nodes covered by a node-disjoint union of edges and a set of allowed odd cycles. For the topic of packings in undirected graphs, see G. Cornuéjols, D. Hartvigsen and W. Pulleyblank [1,3,2,4], and M. Loebl and S. Poljak [14].

Let $G=(V,E)$ be a directed graph. A *cycle (path)* is the arc-set of a closed (unclosed) directed walk without repetition of arcs or nodes – the emptyset is a path on a single node. A *path-cycle-matching* is the arc-set $M\subseteq E$ of a subgraph of G which is a node disjoint union of paths and cycles. We call an arc $e=uv\in E$ *symmetric*, if $vu\in E$, otherwise e is *asymmetric*. A cycle is *even*, if it consists of an even number of arcs; it is *asymmetric*, whenever it has at least one asymmetric arc. A loop is considered to be an odd cycle of length one, and to be symmetric. Given a directed graph G, let $\mathcal{H}$ be a family of cycles in G which contains all the even cycles, all the asymmetric cycles, and a set (possibly empty) of symmetric odd cycles. Such a set $\mathcal{H}$ is called *symmetric* for G. (Notice, here we have the freedom to drop or include some symmetric odd cycles in $\mathcal{H}$.) An *$\mathcal{H}$-matching* is a path-cycle-matching such that all of its cycles are in $\mathcal{H}$. Let $\nu^{\mathcal{H}}(G)$ denote the maximum cardinality of an $\mathcal{H}$-matching. We consider the problem of determining $\nu^{\mathcal{H}}(G)$ in view of formula (1).

Let us denote by $\mathcal{H}_{\text{even}}$ the set of even cycles in G. So $\mathcal{H}_{\text{even}}$ is symmetric for G if and only if each odd cycle of G is symmetric, let us call such graphs "odd-cycle-symmetric". A directed graph G is called "weakly symmetric" if all edges inside a strongly connected component of G are symmetric. Weakly symmetric graphs are a strict subclass of odd-cycle-symmetric graphs. W. H. Cunningham and J. F. Geelen considered the maximum even factor problem – which is equivalent to the notion $\mathcal{H}_{\text{even}}$-matchings in our terminology. By extending their linear algebraic methods developed for path-matchings [6], they proved a min-max formula for maximum even factors in weakly symmetric graphs and gave an algorithm based on computing the rank of the Tutte-matrix [5]. We mention that T. Király and M. Makai recently gave a TDI description of a polyhedron corresponding to a projective image of even factors in a weakly symmetric graph [12].

In [17] a combinatorial proof was given for a formula on $\nu^{\mathcal{H}_{\text{even}}}(G)$ in weakly symmetric graphs. By extending the methods from [9,17], in this paper we consider the $\mathcal{H}$-matching problem. We prove a min-max relation together with a structural and matroidal description. We will demonstrate a relation with the Tutte-matrix of a directed graph; it seems to us that our approach gives a more general setting than the algebraic approach.

Some further definitions regarding formula (1): $N_G^+(X) := \{x \in V - X : \exists y \in X, yx \in E\}$. We say that some cycles of $\mathcal{H}$ *internally cover* a vertex-set $C \subseteq V$ if these cycles are node disjoint and the union of their node-sets is exactly C. The node-set of a directed graph can be partitioned into strongly connected components, the contraction of which leaves an acyclic graph. A strongly connected component will be called a *source-component* if it corresponds to a source-node in the contracted graph (a source-node is a node with no entering arc). For a node-set $X \subseteq V$, let $G[X]$ denote the subgraph of G spanned by X. $sc^{\mathcal{H}}G[X]$ denotes the number of those source components C in $G[X]$ that cannot be internally covered by some cycles in $\mathcal{H}$.

The main result of this paper is the following min-max formula for the maximum cardinality of an $\mathcal{H}$-matching.

Theorem 1.1. *If $\mathcal{H}$ is symmetric for directed graph $G = (V, E)$, then*

$$\nu^{\mathcal{H}}(G) = \min_{X \subseteq V} \{|V| + |N_G^+(X)| - sc^{\mathcal{H}}G[X]\}. \tag{1}$$

To see the easy part of the proof we show that for any $\mathcal{H}$-matching M and any set $X \subseteq V$ inequality $|M| \leq |V| + |N_G^+(X)| - sc^{\mathcal{H}}G[X]$ holds. This implies that the left hand side is at most the right hand side in formula (1). We get this inequality as the sum of the inequalities below:

$$|i_G(X) \cap M| \leq |X| - sc^{\mathcal{H}}G[X], \tag{2}$$

$$|\delta_G(X) \cap M| \leq |N_G^+(X)|, \tag{3}$$

$$|(i_G(V - X) \cup \delta_G(V - X)) \cap M| \leq |V| - |X|, \tag{4}$$

where $i_G(X)$ denotes the set of the arcs of G with both ends in X and $\delta_G(X)$ denotes the set of the arcs of G with tail in X and head in $V - X$.

We demonstrate the theorem with the following example.

Let $\mathcal{H}$ be the set of even cycles and asymmetric cycles for the graph G in Figure 1, so $\mathcal{H}$ is symmetric for G. The sets X and $N_G^+(X)$ are indicated in the figure, the dashed parts are the source components of $G[X]$. Here $|V| + |N_G^+(X)| - sc^{\mathcal{H}}G[X] = 16 + 1 - 3 = 14$, and it is easy to find an $\mathcal{H}$-matching of size 14.

Theorem 1.1 heavily relies on the assumption that $\mathcal{H}$ is symmetric for G, as the following example shows. For graph G in Figure 2 and $\mathcal{H} = \mathcal{H}_{\text{even}} = \emptyset$ we have $\nu^{\mathcal{H}}(G) = 7$, while in the right hand side of (1) we have 8, achieved by $X = V$. In fact, the $\mathcal{H}_{\text{even}}$-matching problem in an arbitrary graph is NP-hard, see Cunningham [5].

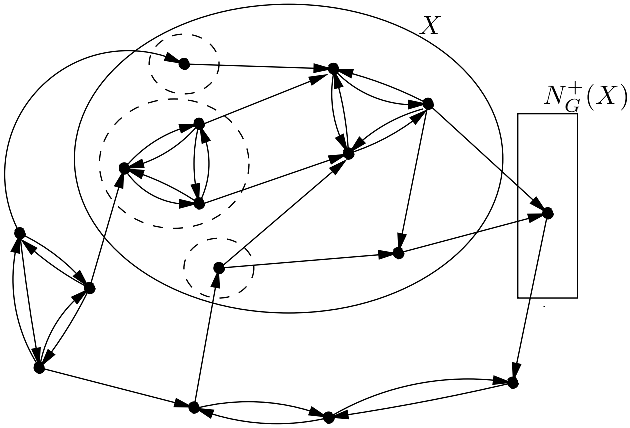

Fig. 1.

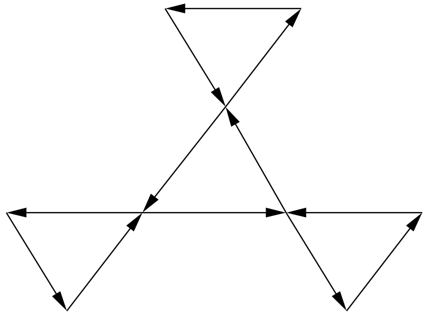

Fig. 2.

2. Preliminaries

For an introduction to matching theory see L. Lovász an M. D. Plummer [16]; in this paper we will make use of the following notions. An undirected graph is called *factor-critical* if the deletion of any node leaves a graph having a perfect matching (i.e. it is perfectly matchable). For the case of directed graphs, *factor-critical* means that all arcs are symmetric, and the underlying undirected graph is factor-critical. Now, $fc^{\mathcal{H}}G[X]$ denotes the number of source components in $G[X]$ which cannot be covered by $\mathcal{H}$-cycles and are factor-critical. Let $Fc^{\mathcal{H}}G[X]$ denote the union of these components. A directed graph is said to be $\mathcal{H}$-critical, if it is factor-critical, and it cannot be covered by $\mathcal{H}$-cycles. Clearly, $fc^{\mathcal{H}}G[X] \leq sc^{\mathcal{H}}G[X]$. The following strengthening of Theorem 1.1 will be easier to prove:

Theorem 2.1. *If $\mathcal{H}$ is symmetric for directed graph $G=(V,E)$, then*

$$\nu^{\mathcal{H}}(G) = \min_{X\subseteq V}\{|V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X]\}. \tag{5}$$

The proof in the paper refers to the following well-known fact from matching theory.

Lemma 2.2. *Suppose $s,t \in V$ are two (not necessarily distinct) nodes of a factor-critical graph $G=(V,E)$. Then there is a path $P(s,t)$ on an even number of edges such that $G[V-V(P)]$ is perfectly matchable. If $s=t$, then P is an empty path.*

Let us mention the following description of the maximum matchings in an undirected graph which was given by T. Gallai [10] and J. Edmonds [7]. In Section 5 we will show a generalization to the setting of $\mathcal{H}$-matchings.

Theorem 2.3 (Edmonds, Gallai). *Let $G' = (V', E')$ be an undirected graph. Let $D' := \{v' \in V'$: there exists a maximum matching in G' exposing $v'\}$.*

1. *A maximum matching covers exactly $|V'|+|\Gamma_{G'}(D')|-c(G'[D'])$ number of nodes, where $c(G'[D'])$ denotes the number of components in $G'[D']$.*
2. *All the components of $G'[D']$ are factor-critical.*

3. Remarks

We mention two previous results which have motivated, and are special cases of Theorem 1.1 and 2.1.

A factorization problem was addressed by Cornuéjols and Hartvigsen in [1], and by Cornuéjols and Pulleyblank in [3]. The so-called triangle-free 2-matching problem was discussed in detail: they gave a formula for the maximum number of nodes that can be covered by a node-disjoint collection of edges and odd cycles of length at least 5. The following related theorem due to M. Loebl and S. Poljak [14] characterizes a factorization problem in undirected graphs, see also J. Szabó and Z. Király [13]:

Theorem 3.1. *Let $G' = (V', E')$ be an undirected graph, and $\mathcal{H}'$ be a set of some (maybe none) odd cycles in G'. Let $\nu^{\mathcal{H}'}(G')$ denote the maximum number of nodes which can be covered by a node-disjoint collection of edges and $\mathcal{H}'$-cycles. Then*

$$\nu^{\mathcal{H}'}(G) = \min_{X\subseteq V} |V| - c^{\mathcal{H}'}(X) + |X|, \tag{6}$$

where $c^{\mathcal{H}'}(X)$ *denotes the number of factor-critical components of* $G-X$ *which cannot be internally covered by edges and* $\mathcal{H}'$*-cycles (i.e.* $\mathcal{H}'$-critical *components).*

If G is a symmetric directed graph, then any family $\mathcal{H}$ containing the even cycles is symmetric for G. Thus, for symmetric directed graphs Theorem 1.1 is equivalent to Theorem 3.1 applied to the underlying undirected graph.

In [17] Pap and Szegő gave a formula for $\nu^{\mathcal{H}_{\text{even}}}(G)$ for any weakly symmetric graph G, that formula is a special case of Theorem 1.1. We also mention that the following theorems can be deduced from the formula in [17] for $\nu^{\mathcal{H}_{\text{even}}}(G)$: Dilworth's theorem on the maximum number of independent elements in a partially ordered set, Menger's theorem on disjoint paths, a theorem of Gallai and Milgram on minimum number of directed path to cover all nodes in a directed graph, a theorem of S. Felsner in [8] on maximum number of arcs in a path-cycle-matching, a formula in [9] for path-matchings. For proofs, see [17].

Much of this research was motivated by the notions path-matching and even factor introduced by Cunningham and Geelen (see [5,6]). They gave good characterizations, as well as an algorithm based on the following algebraic method. For a directed graph $G=(V,E)$, we define a $V\times V$ matrix $M=M(G)$ of commuting, algebraically independent indeterminates:

$$\begin{array}{ll} M_{u,v} := 0 & \text{if } uv \notin E(G), \\ M_{u,v} := x_{u,v} & \text{if } uv \in E(G) \text{ and } vu \notin E(G), \\ M_{u,v} := x_{u,v} \text{ and } M_{v,u} := -x_{u,v} & \text{if } uv \in E(G) \text{ and } vu \in E(G). \end{array}$$

Take an undirected graph G', let G'' be constructed from G' by replacing each edge uv in $E(G')$ by arcs uv and vu. The matching number of an undirected graph G' can be determined as half the rank of a matrix $M(G'')$. Thus, for symmetric directed graphs we have a combinatorial description for the rank of M. More generally, in case of G being weakly symmetric we have $rk(M)=\nu^{\mathcal{H}_{\text{even}}}(G)$.

Geelen [11] discovered an algorithm to calculate the rank $rk(M)$ of matrix M for any directed graph G. Since the rank is equal to the rank for some evaluation of the indeterminates, one has to find a nice evaluation. We get a randomized algorithm – due to L. Lovász [15] – if we put uniformly distributed independent values from $\{1,\dots,|V|\}$. Geelen's algorithm is a derandomization, which gives a deterministic algorithm to calculate the maximum cardinality of an even factor, it can be used to determine a maximum even factor, too.

Let G be an arbitrary graph, and let $\mathcal{H}_{\text{even and asym.}}$ be the collection of even cycles and asymmetric cycles. It is easy to see that

$$rk(M) = \nu^{\mathcal{H}_{\text{even and asym.}}}(G).$$

Geelen's method gives a polynomial algorithm to compute this number. Theorem 1.1 gives a formula for a more general case: the only constraint on set $\mathcal{H}$ is $\mathcal{H}_{\text{even and asym.}} \subseteq \mathcal{H}$. One does not expect to have a polynomial algorithm in this generality. Theorem 1.1 is not a good characterization, since it would require being able to decide whether some $G[C]$ can be internally covered by some cycles of $\mathcal{H}$. Suppose we have an oracle to decide for any factor-critical subgraph $G[C]$ if it can be internally covered by some cycles of $\mathcal{H}$, and it returns a covering if any. So Theorem 2.1 is a good characterization, and we may hope for a polynomial time algorithm.

Consider the following striking statement given by Cornuéjols, Hartvigsen and Pulleyblank [2]. If a factor-critical (undirected) graph can be internally covered by edges and some cycles of $\mathcal{H}'$, then there is a covering using exactly one odd cycle of $\mathcal{H}'$. Thus, Theorem 2.1 is a good characterization in the case when the odd cycles of $\mathcal{H}$ can be listed in polynomial time. This is the case if the length of odd cycles in $\mathcal{H}$ is bounded. We get another special case when an oracle is known by the following lemma:

Lemma 3.2 (Cornuéjols, Pulleyblank [3]). *Let G' be an undirected graph and suppose $\mathcal{H}'$ is a set of odd cycles in G' which contains all odd cycles longer than 3. Then G' is $\mathcal{H}'$-critical if and only if it is a triangle cluster of triangles not in $\mathcal{H}'$.*

A *triangle cluster* is the single node graph, and each graph we get by the following operation: choose an old node a, add new nodes b, c and arcs ab, bc, ca to the graph. A directed graph is called a triangle cluster, if it is symmetric, and the underlying undirected graph is a triangle cluster.

Theorem 3.3. *Suppose $G = (V, E)$ is a directed graph, such that each directed cycle of three arcs is symmetric. Then the maximum number of arcs in a path-cycle-matching without three-arc cycles is*

$$\min_{X \subseteq V} |V| + |N_G^+(X)| - tc\, G[X] \tag{7}$$

where $tc\, G[X]$ is the number of source components of $G[X]$ which are triangle clusters.

It is easy to see that $tc\, G[X]$ can be determined in polynomial time. In fact an oracle to determine $fc^{\mathcal{H}} G[X]$ exists in several cases. For any k there is a polynomial time recognition algorithm for $\mathcal{H}$-critical graphs if either

1. $\mathcal{H}$ contains all odd cycles longer than k, or
2. $\mathcal{H}$ contains no odd cycles longer than k.

For a proof see G. Cornuéjols and W. R. Pulleyblank [4].

4. Proof

Theorem 3.1 can be derived from standard results on matchings, so one may expect that Theorem 2.1 can be derived from results on even factors. Here we need to point out that there is a great obstacle to this, see Section 5 for details. Instead, we extend the proof in [17] to prove Theorem 2.1.

The proof of Theorem 2.1 will be presented in the following structure: A dividing procedure is presented in CASE 4 which gives two smaller graphs, and proves the formula by induction for most pairs $G,\mathcal{H}$. Cases where the dividing procedure does not lead to graphs with less edges will be discussed in the first three cases.

In a subgraph G' of G, if we use the letter $\mathcal{H}$, that means the truncation of $\mathcal{H}$ to those cycles of G which are also cycles in G'. This is legitimate since in this sense $\mathcal{H}$ will be symmetric for G'.

A set X is called *minimizing* if it minimizes $|V|+|N_G^+(X)|-fc^{\mathcal{H}}G[X]$. A set X is called *trivial* if one of the following holds:

(i) The source components of $G[X]$ are single nodes, $V=X\cup N_G^+(X)$ and there is no arc uv such that $u\in N_G^+(X)$.
(ii) X is a stable set in G, and there is no arc uv such that $u\in X$ and $v\in V-X$.

We have already proved in the introduction that the left hand side is at most the right hand side in formula (1). The proof that there is a set X and an $\mathcal{H}$-matching K such that $|K|=|V|+|N_G^+(X)|-fc^{\mathcal{H}}G[X]$ goes by induction on $|E|+|V|$. Without loss of generality we may assume that G is weakly connected, that is, its underlying undirected graph is connected.

Observation 4.1. *$X=V$ is the only possibility for a minimizing set of type* (i)*.*

Proof. If $X\neq V$ is a minimizing set of type (i), then since G is weakly connected, $|N_G^+(X)|>0$. Thus $|V|+|N_G^+(X)|-fc^{\mathcal{H}}G[X]>|V|-fc^{\mathcal{H}}G[V]$, a contradiction. ∎

CASE 1. G is symmetric.

In this case formula (1) follows from Theorem 3.1.

Thus from now on we may assume that G is not symmetric. For better reading in the forthcoming part, we use $\tau_G^{\mathcal{H}}(X) := |V| + |N_G^+(X)| - fc^{\mathcal{H}} G[X]$ for the *value* of set X in G. Let $\tau_G^{\mathcal{H}} := \min_{X \subseteq V} \tau_G^{\mathcal{H}}(X)$ be the value of a minimizing set in G. Let $uv = e \in E$ be an arc such that $vu \notin E$. We observe that $\mathcal{H}$ is symmetric for $G - e$. For any set X

$$\tau_{G-e}^{\mathcal{H}}(X) \leq \tau_G^{\mathcal{H}}(X) \leq \tau_{G-e}^{\mathcal{H}}(X) + 1 \tag{8}$$

with $\tau_G^{\mathcal{H}}(X) = \tau_{G-e}^{\mathcal{H}}(X) + 1$ if and only if for $e = uv$ either

A) $u \in X$ and $v \in V - X - N_{G-e}^+(X)$ or

B) $u \in X$ and $v \in Fc^{\mathcal{H}}(G-e)[X]$.

CASE 2. There exits a trivial minimizing set of type (ii).

Claim 4.2. *If there is a trivial minimizing set of type (ii), then the formula holds.*

Proof. Take an arc $e = uv$ such that $vu \notin E$. If $\tau_{G-e}^{\mathcal{H}} = \tau_G^{\mathcal{H}}$, then we are done by induction. Otherwise (8) implies that for any minimizing set X in G

$$\tau_G^{\mathcal{H}} = \tau_G^{\mathcal{H}}(X) = \tau_{G-e}^{\mathcal{H}}(X) + 1.$$

Take a minimizing set X in G. By assumption, X is a trivial set in G. Arc e accords to A) or B), so X cannot be of type (ii), a contradiction. ∎

CASE 3. Every minimizing set is trivial.

Take a minimizing set X in G. By assumption, X is a trivial set in G, and by Claim 4.2 it must be of type (i). By Claim 4.1 $X = V$.

Now $X = V$ is a minimizing set of type (i), so there must be at least one source-node in G. Take arc $e' = u'v'$ such that $\{u'\}$ is a source-node in G. $e' = u'v'$ must be of type B), and then $V - u'$ is a minimizing set in G. By assumption, $V - u'$ is tight and by Observation 4.1 it can only be of type (ii), a contradiction.

CASE 4. In any other case, let us consider a minimal nontrivial minimizing set X.

Claim 4.3. *Each source component of $G[X]$ is $\mathcal{H}$-critical.*

Proof. If a source component $G[C]$ of $G[X]$ can be internally covered with cycles in $\mathcal{H}$, then $X - C$ is also a minimizing set. If $X - C$ is nontrivial, then this contradicts the minimality of X. Thus X–C is trivial, by Observation 4.1 $X - C$ is of type (ii), and by Claim 4.2 we are done.

Suppose a source component $G[C]$ of $G[X]$ cannot be internally covered, but is not factor-critical. $C \neq V$, thus $G[C]$ has less arcs, than G has. Then by induction there is a subset $Y \subseteq C$ with value $\tau^{\mathcal{H}}_{G[C]}(Y) \leq |C|-1$. Since $G[C]$ is not factor-critical, $\tau^{\mathcal{H}}_{G[C]}(C) = |C|$ and $\tau^{\mathcal{H}}_{G[C]}(\emptyset) = |C|$, thus Y is a proper nonempty subset of C. It is easy to see that $X - C \cup Y$ is a minimizing set in G. C is strongly connected, therefore $X - C \cup Y$ must be entered as well as left by arcs of G. Then $X - C \cup Y$ is a nontrivial minimizing set, a contradiction. ∎

Delete the arc-set $F := \{uv \in E : u \in V - X,\ v \in N^+_G(X)\}$ and contract each component of $Fc_G(X)$ to a node. Let $G_Q = (V_Q, E_Q)$ denote the graph obtained this way. Q denotes the set of new nodes, define $X_Q := X - Fc^{\mathcal{H}}G[X] \cup Q$.

Let $G_1 = (V_1, E_1)$ denote the graph having node set $V_1 := X_Q \cup N^+_G(X)$ and arc set $E_1 := \{uv \in E_Q : u \in X_Q\}$.

Let $G_2 = (V_2, E_2)$ denote the graph having node set $V_2 := Q \cup (V_Q - X_Q)$ and arc set $E_2 := \{uv \in E_Q : v \in V_2 - N^+_G(X)\}$.

The cycles in G_1 and G_2 are also cycles in G. When using $\mathcal{H}$ for G_i, it stands for the truncation of $\mathcal{H}$ to E_i. Clearly, $\mathcal{H}$ is symmetric for G_1 and G_2. Since X is nontrivial, $|E_1| < |E|$ and $|E_2| < |E|$.

Claim 4.4. *Suppose K_1, K_2 are $\mathcal{H}$-matchings in G_1, G_2, respectively. Then G has an $\mathcal{H}$-matching K with cardinality $|K| = |K_1| + |K_2| + (|Fc^{\mathcal{H}}G[X]| - fc^{\mathcal{H}}G[X])$.*

Proof. Let K' denote the set of arcs of G corresponding to $K_1 \cup K_2$. We claim that K' can be completed in G so that it has the desired cardinality. To this end let C denote a component of $Fc^{\mathcal{H}}G[X]$, and let c denote its corresponding node in G_Q. By Claim 4.3, C is $\mathcal{H}$-critical.

K' has at most one arc in $\delta_G(C)$: choose $t \in C$ as the tail of this arc if present, otherwise choose t arbitrarily. K' has at most one arc in $\varrho_G(C)$: choose $s \in C$ as the head of this arc if present, otherwise choose s arbitrarily. By Lemma 2.2, there is a path-cycle-matching K_C in $G[C]$ of size $|C| - 1$, consisting of two-arc cycles and an $s-t$ path on an even number of arcs.

$K := K' \cup \bigcup_{c \in Q} K_C$ is a path-cycle-matching with cardinality $|K| = |K_1 \cup K_2| + (|Fc^{\mathcal{H}}G[X]| - |Q|)$. We only have to check, if the cycles traversing $Fc^{\mathcal{H}}G[X]$ are in $\mathcal{H}$:

Suppose that a cycle $W \subseteq K$ is not in $\mathcal{H}$. Let W_Q be the cycle in G_Q corresponding to W. All arcs in W are symmetric in G, hence W_Q has no arc from Q to $X - Q$, and from $V - X - N^+_G(X)$ to X. By the definition of G_2, W_Q has no arc from $V - X$ to $N^+_G(X) \cup (X - Q)$. Then W_Q can only be a cycle alternating between Q and $N^+_G(X)$, thus W_Q, W are even cycles, W is in $\mathcal{H}$. ∎

Claim 4.5. *G_1 has an $\mathcal{H}$-matching K_1 with cardinality $|V_1|-fc^{\mathcal{H}}G[X]$.*

Proof. By induction, it is enough to prove that $\tau_{G_1}(Y)\geq|V_1|-fc^{\mathcal{H}}G[X]$ holds for all $Y\subseteq V_1$.

$\tau_{G_1}(Y)\geq\tau_{G_1}(Y\cup N_G^+(X))$, hence we suppose that $N_G^+(X)\subseteq Y\subseteq V_1$. Let $S:=\{v\in N_G^+(X)$: there is no arc uv with $u\in Y-N_G^+(X)\}$.

We have $N_{G_1}^+(X_Q\cap Y)=N_{G_1}^+(Y)\cup(N_G^+(X)-S)$, thus

$$|N_{G_1}^+(X_Q\cap Y)|\leq|N_{G_1}^+(Y)|+|N_G^+(X)|-|S|, \tag{9}$$

$$fc^{\mathcal{H}}G_1[Y]-|S|=fc^{\mathcal{H}}G_1[X_Q\cap Y]. \tag{10}$$

Let Y_G denote the set we get from Y after replacing the nodes of $Y\cap Q$ by the corresponding nodes in G. Since X is a minimizing set in G,

$$|V|+|N_G^+(X)|-fc^{\mathcal{H}}G[X]\leq|V|+|N_G^+(X\cap Y_G)|-fc^{\mathcal{H}}G[X\cap Y_G]. \tag{11}$$

It is easy to see, that $fc^{\mathcal{H}}G[X]=|Q|=fc^{\mathcal{H}}G_1[X_Q]$, $N_G^+(X\cap Y_G)=N_{G_1}^+(X_Q\cap Y)$, and $fc^{\mathcal{H}}G[X\cap Y_G]=fc^{\mathcal{H}}G_1[X_Q\cap Y]$. Then by inequality (11) we get

$$|N_G^+(X)|-fc^{\mathcal{H}}G_1[X_Q]\leq|N_{G_1}^+(X_Q\cap Y)|-fc^{\mathcal{H}}G_1[X_Q\cap Y]. \tag{12}$$

By adding up (9), (10) and (12)

$$fc^{\mathcal{H}}G_1[Y]-fc^{\mathcal{H}}G_1[X_Q]\leq|N_{G_1}^+(Y)|. \tag{13}$$

Thus,

$$|V_1|-fc^{\mathcal{H}}G[X]=|V_1|-fc^{\mathcal{H}}G[X_Q]\leq|V_1|+|N_{G_1}^+(Y)|-fc^{\mathcal{H}}G_1[Y] =\tau_{G_1}(Y). \tag{14}$$ ∎

Claim 4.6. *G_2 has an $\mathcal{H}$-matching K_2 with cardinality $|V_2|-|Q|$.*

Proof. By induction, it is enough to prove that $\tau_{G_2}(Z)\geq|V_Q|-|X_Q|$ holds for all $Z\subseteq V_2$.

$\tau_{G_2}(Z)\geq\tau_{G_2}(Z\cup Q)$, hence we suppose that $Q\subseteq Z\subseteq V_2$. Let Z_G denote the set we get from Z after replacing the nodes of Q by the corresponding nodes in G.

We have $N_G^+(X\cup Z_G)=(N_G^+(X)-(Z\cap N_G^+(X)))\cup N_{G_2}^+(Z)$, thus

$$|N_G^+(X\cup Z_G)|=|N_G^+(X)|-|Z\cap N_G^+(X)|+|N_{G_2}^+(Z)|. \tag{15}$$

Since X is tight in G,

$$|V|+|N_G^+(X)|-fc^{\mathcal{H}}G[X]\leq|V|+|N_G^+(X\cup Z_G)|-fc^{\mathcal{H}}G[X\cup Z_G]. \tag{16}$$

Now we prove inequality (17). Consider the $\mathcal{H}$-critical source components of $G_2[Z]$. These are all the nodes in $Z \cap N_G^+(X)$ as single node components and some other components disjoint from $N_G^+(X)$. The latter type components give $\mathcal{H}$-critical source components of $G[X \cup Z_G]$, too. This proves

$$fc^{\mathcal{H}}G_2[Z] - |Z \cap N_G^+(X)| \le fc^{\mathcal{H}}G[X \cup Z_G]. \tag{17}$$

By adding up (15), (16) and (17)

$$fc^{\mathcal{H}}G_2[Z] - |Q| = fc^{\mathcal{H}}G_2[Z] - fc^{\mathcal{H}}G[X] \le |N_{G_2}^+(Z)|. \tag{18}$$

Thus,

$$|V_2| - |Q| \le |V_2| + |N_{G_2}^+(Z)| - fc^{\mathcal{H}}G_2[Z]. \qquad \blacksquare$$

By Claims 4.4, 4.5 and 4.6, G has an $\mathcal{H}$-matching K of cardinality $|V| + |N_G^+(X)| - fc^{\mathcal{H}}G[X]$. This completes the proof in CASE 4. ■ ■

5. Structural description

Theorem 5.1 (Structure Theorem). *Suppose $\mathcal{H}$ is symmetric for directed graph $G=(V,E)$. Let $D := D(\mathcal{H}) := \{v \in V :$ there exists a maximum $\mathcal{H}$-matching K such that $\delta_K(v)=0\}$.*

1. *$\nu(G) = |V| + (|N_G^+(D)| - fc^{\mathcal{H}}G[D])$, and*
2. *the source components of $G[D]$ are $\mathcal{H}$-critical.*

Proof. Let X be a minimizing set such that $|X|$ is minimum. We are going to prove that $X = D$. It follows from Claim 4.3 that each source component of $G[X]$ is $\mathcal{H}$-critical. First we prove that $D \subseteq X$. Take any node $v \in D$. Let K_v be an even factor of size $|K_v| = \tau_G = \tau_G(X)$, with $\delta_{K_v}(v) = 0$. For $K = K_v$, we must have equality in (2)–(4). From equality in (4) we get that $v \notin V - X$.

Now we prove $X \subseteq D$. Consider G_Q, G_1 and G_2 which were defined for any minimizing set in the proof of Theorem 2.1. By Claims 4.4 and 4.6, the following claim finishes the proof of Theorem 5.1.

Claim 5.2. *For any $v \in X_Q$, there is an $\mathcal{H}$-matching K_1 with cardinality $|V_1| - fc^{\mathcal{H}}G[X]$ such that $\delta_{K_1}(v) = 0$.*

Proof. Let G_1' denote the graph obtained from G_1 by deleting the arcs coming out of v. Clearly, $\mathcal{H}$ is symmetric for G_1'. We have to prove that there is a $\mathcal{H}$-matching in G_1' of cardinality $|V_1| - fc^{\mathcal{H}}G[X]$.

We are going to prove, that $\tau_{G_1}(Y) \ge |V_1| - fc^{\mathcal{H}}G[X] + 1$ for any $Y \subseteq V_1 - v$. Then by Theorem 2.1 we will be done, since $\tau_{G_1'}(Y + v) \ge \tau_{G_1}(Y) - 1$ for any set $Y \subseteq V_1 - v$.

If $Y \subseteq V_1 - v$, then $\tau_{G_1}(Y) \geq \tau_{G_1}(Y \cup N_G^+(X))$, hence we suppose, that $N_G^+(X) \subseteq Y \subseteq V_1 - v$. Let $S := \{w \in N_G^+(X)$: there is no arc uw with $u \in Y - N_G^+(X)\}$. We have $N_{G_1}^+(X_Q \cap Y) = N_{G_1}^+(Y) \cup (N_G^+(X) - S)$, thus

$$|N_{G_1}^+(X_Q \cap Y)| \leq |N_{G_1}^+(Y)| + |N_G^+(X)| - |S|, \tag{19}$$

$$fc^{\mathcal{H}} G_1[Y] - |S| = fc^{\mathcal{H}} G_1[X_Q \cap Y]. \tag{20}$$

Let Y_G denote the resulting set after replacing the nodes of $Y \cap Q$ by the corresponding source components of $G[X]$ in Y. Since X is a minimum minimizing set in G,

$$|V| + |N_G^+(X)| - fc^{\mathcal{H}} G[X] + 1 \leq |V| + |N_G^+(X \cap Y_G)| - fc^{\mathcal{H}} G[X \cap Y_G]. \tag{21}$$

It is easy to see, that $fc^{\mathcal{H}} G[X \cap Y_G] = fc^{\mathcal{H}} G_1[X_Q \cap Y]$, then by inequality (21):

$$|N_G^+(X)| - fc^{\mathcal{H}} G[X] + 1 \leq |N_{G_1}^+(X_Q \cap Y)| - fc^{\mathcal{H}} G_1[X_Q \cap Y]. \tag{22}$$

By adding up (19), (20) and (22) we get

$$fc^{\mathcal{H}} G_1[Y] - fc^{\mathcal{H}} G[X] + 1 \leq |N_{G_1}^+(Y)|,$$

thus,

$$|V_1| - fc^{\mathcal{H}} G[X] + 1 \leq |V_1| + |N_{G_1}^+(Y)| - fc^{\mathcal{H}} G_1[Y] = \tau_{G_1}(Y). \quad \blacksquare$$

Let us make a remark on the set $D = D(\mathcal{H})$ of Theorem 5.1. If we apply this theorem for a symmetric graph G and $\mathcal{H} = \mathcal{H}_{\text{even}}$, then we get back to the Edmonds–Gallai decomposition. If G is symmetric, but we consider an arbitrary set $\mathcal{H} \supseteq \mathcal{H}_{\text{even}}$, then we get back to Theorem 3.1. In fact, Theorem 3.1 can be proved using a reduction to properties of the Edmonds–Gallai decomposition, see Loebl, Poljak [14]. We also get that – in the case when G is symmetric –

$$D(\mathcal{H}) \subseteq D(\mathcal{H}_{\text{even}}).$$

One would expect that the analogue statement $D(\mathcal{H}) \subseteq D(\mathcal{H}_{\text{even or asym.}})$ holds whenever $\mathcal{H}$ is symmetric for G. This is not true in general, a small counterexample is given by the directed graph on nodes $V = \{a, b\}$, arcs $E = \{ab, bb\}$ with $\mathcal{H} = \{bb\}$. There $\nu^{\mathcal{H}}(G) = \nu^{\mathcal{H}_{\text{even or asym.}}}(G) = 1$ and $D(\mathcal{H}) = \{a, b\}$ and $D(\mathcal{H}_{\text{even or asym.}}) = \{b\}$. This is a reason why we could not reduce the main Theorem 2.1 to results on even factors.

6. Matroidal description

In an undirected graph, the system of node-sets which can be covered by a matching gives the independent sets of the so-called matching matroid. The Tutte-matrix gives a linear representation of the matching matroid. As a generalization, we give a matroid corresponding to a pair $G,\mathcal{H}$ where $\mathcal{H}$ is symmetric for G. Finding a linear representation of this matroid is an open problem.

For an $\mathcal{H}$-matching M define $V_+(M):=\{v\in V:|\delta_M(v)|=1\}$.

Theorem 6.1. *Suppose $\mathcal{H}$ is symmetric for directed graph $G=(V,E)$. The following family defines the independent sets of a matroid:*

$$\mathcal{I}(G,\mathcal{H}) := \{I \subseteq V : \text{ there is a maximum } \mathcal{H}\text{-matching } M \text{ such that } I \subseteq V_+(M)\}. \tag{23}$$

To show a version of the matroid exchange axiom, it suffices to prove the following lemma:

Lemma 6.2. *Suppose M_1 and M_2 are $\mathcal{H}$-matchings with $|M_1|<|M_2|$. Then there is an $\mathcal{H}$-matching M_1' such that $V_+(M_1)\subset V_+(M_1')$ and $V_+(M_1')-V_+(M_1)\subseteq V_+(M_2)$.*

Proof. Consider the graph G' we get by deleting all arcs $uv\in E(G)$ for $u\in V-(V_+(M_1)\cup V_+(M_2))$. Clearly, $\mathcal{H}$ is symmetric for G', and M_1, M_2 are $\mathcal{H}$-matchings in G'.

Let $k:=|V_+(M_2)-V_+(M_1)|-1$. We construct a graph G'' from G' as follows. We add a set U of k new nodes, that is $V(G''):=V(G')\cup U$. We add $k\cdot(k+1)$ new arcs, each possible arc uv for $u\in V_+(M_2)-V_+(M_1)$ and $v\in U$. Then $\mathcal{H}$ is symmetric for G''. We are going to prove that there is an $\mathcal{H}$-matching M in G'' with $|M_1|+k+1=|V_+(M_2)\cup V_+(M_1)|$ arcs. If there is such an $M\subseteq E(G'')$, then $M_1':=M\cap E(G)$ will do.

Suppose for a contradiction that $\nu^{\mathcal{H}}(G'')\le |M_1|+k$. Then $\nu^{\mathcal{H}}(G'')=|M_1|+k$, since we can add to M_1 k disjoint arcs from $V_+(M_2)-V_+(M_1)$ to U. By Theorem 5.1 we get for $D''=D(G'')$

$$|M_1|+k=|V(G'')|-fc^{\mathcal{H}}G''[D'']+|N^+_{G''}(D'')|. \tag{24}$$

Since there is no arc leaving any node in U we get $U\subseteq D''$, thus $N^+_{G''}(D'')=N^+_{G'}(D''-U)$. For each node v in $V_+(M_2)-V_+(M_1)$ one can construct an $\mathcal{H}$-matching in G'' of $|M_1|+k$ arcs with no arc leaving v, thus $V_+(M_2)-$

$V_+(M_1)\subseteq D''$. Then the source-components in $G''[D'']$ are disjoint from U, thus $fc^{\mathcal{H}}G''[D'']=fc^{\mathcal{H}}G'[D''-U]$.

$$\tau^{\mathcal{H}}_{G'}(D''-U)=|V(G')|-fc^{\mathcal{H}}G'[D''-U]+|N^+_{G'}(D''-U)|= \\ =|V(G'')|-k-fc^{\mathcal{H}}G''[D'']+|N^+_{G''}(D'')|=\nu^{\mathcal{H}}(G'')-k=|M_1|. \tag{25}$$

$\tau^{\mathcal{H}}_{G'}(X)$ is an upper bound for the cardinality of any $\mathcal{H}$-matching in G', then there cannot be any greater than $|M_1|$. This is in contradiction with the existence of M_2. ∎

7. Conclusions

The paper gives a general concept for matching in a directed graph which is a natural generalization of well-known theorems and solves some new cases, too. A compact formula and description could be given as a direct extension of earlier results. The descriptions given here are tractable if an oracle for recognizing $\mathcal{H}$-critical graphs exists. Although this oracle exists in many cases, we have some open questions. First, give a polynomial time algorithm to determine $\nu^{\mathcal{H}}(G)$ based on this oracle. Second, can the matroid $\mathcal{I}(G,\mathcal{H})$ be represented linearly? In other words, is it possible to extend Cunningham and Geelen's algebraic approach for even factors to $\mathcal{H}$-matchings?

Furthermore, in the undirected case a description could be given for hypomatchings – factorizations involving arbitrary factor-critical graphs, not only odd cycles. Is there a factorization problem in directed graphs which also generalizes this problem?

Acknowledgements

We thank Zoltán Király, Márton Makai and Jácint Szabó for fruitful discussions on the topic. We are very grateful to the anonymous referees for their helpful suggestions.

References

[1] G. Cornuéjols and D. Hartvigsen: An extension of matching theory, *Journal of Combinatorial Theory Ser. B* **40** (1986), 285–296.

[2] G. Cornuéjols, D. Hartvigsen and W. Pulleyblank: *Packing subgraphs in a graph*, Op. Res. Letters, (1982), 139–143.

[3] G. Cornuéjols and W. Pulleyblank: A matching problem with side conditions, *Discrete Mathematics* **29** (1980), 135–159.

[4] G. Cornuéjols and W. Pulleyblank: Critical graphs, matchings and tours or a hierarchy of the travelling salesman problem, *Combinatorica* **3(1)** (1983), 35–52.
[5] W. H. Cunningham: Matching, Matroids and Extensions, *Math. Program. Ser. B* **91(3)** (2002), 515–542.
[6] W. H. Cunningham and J. F. Geelen: The Optimal Path-Matching Problem, *Combinatorica* **17(3)** (1997), 315–336.
[7] J. Edmonds: Paths, trees, and flowers, *Canadian Journal of Mathematics* **17** (1965), 449–467.
[8] S. Felsner: Orthogonal structures in directed graphs, *Journal of Combinatorial Theory Ser. B* **57(2)** (1993), 309–321.
[9] A. Frank and L. Szegő: A Note on the Path-Matching Formula, *J. of Graph Theory* **41(2)** (2002), 110–119.
[10] T. Gallai: Maximale Systeme unabhängiger Kanten, *A Magyar Tudományos Akadémia Matematika Kutatóintézetének Közleményei* **9** (1964), 401–413.
[11] J. Geelen: *An algebraic approach to matching problems*, manuscript.
[12] T. Király and M. Makai: On polyhedra related to even factors, *Proceedings of 10th International IPCO Conference, D. Bienstock, G. Nemhauser (eds.)*, Lecture Notes in Computer Science, Springer, 2004, 416–430.
[13] Z. Király and J. Szabó: *Generalized induced factor problems*, Egres Technical Report, 2002.
[14] M. Loebl and S. Poljak: Efficient Subgraph Packing, *Journal of Combinatorial Theory Ser. B* **59** (1993), 106–121.
[15] L. Lovász: On determinants, matchings and random algorithms, in *Fundamentals of Computational Theory (L. Budach, ed.)*, Akademie-Verlag, Berlin, 1979, 565–574.
[16] L. Lovász and M. D. Plummer: *Matching Theory*, Akadémiai Kiadó, Budapest, 1986.
[17] Gy. Pap and L. Szegő: On the Maximum Even Factor in Weakly Symmetric Graphs, *Journal of Combinatorial Theory Ser. B* **91(2)** (2004), 201–213.

Gyula Pap

Department of Operations Research
Eötvös University
Pázmány Péter sétány 1/C
H-1117 Budapest
Hungary
gyuszko@cs.elte.hu

László Szegő

Department of Operations Research
Eötvös University
Pázmány Péter sétány 1/C
H-1117 Budapest
Hungary
szego@cs.elte.hu